J Glob Optim (2010) 46:503–508
DOI 10.1007/s10898-009-9434-y

On the mean value theorem for semidifferentiable functions

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Received: 6 May 2009 / Accepted: 7 May 2009 / Published online: 21 May 2009

Abstract The G-semidifferentiability concept was introduced in the reference Giannessi (J Optim Theory Appl 60:191–241, 1989) and it furnishes a general scheme for treating generalized derivatives. We prove that, in this context, it is possible to obtain a mean value theorem. By exploiting this result we deduce conditions for a function to be lipschitzian, C-decreasing or quasiconvex.

Keywords Semidifferentiable functions · Mean value theorem

1 Introduction

In 1989, Giannessi wrote a paper [1] which established a very general necessary optimality condition for extremum problems. It was proved within a class of semidifferentiable functions introduced there. In that paper he also proved that the class of semidifferentiable functions embraced several classic types of functions as convex functions, differentiable functions and even some discontinuous functions. Some other papers have been published in this field (see for instance [2,3] and references therein); they proved relationships with other generalized derivatives, calculus rules for this class of functions, generalization to the infinite dimensional case and so on. With this note we want to put new emphasis to this concept by showing a mean value theorem for semidifferentiable functions and we want to stress the fact that, following this scheme, new possibilities of research can be offered in other fields of optimization like descent methods, necessary or sufficient optimality conditions and error bound theory.

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We conclude this introduction recalling the main definitions. We denote by $\mathcal{G}$ the set of all positively homogeneous functions defined on $\mathbb{R}^n$, namely

$$\mathcal{G} = \{g : \mathbb{R}^n \to \mathbb{R} \ : \ g(ty) = tg(y) \quad \text{for all } y \in \mathbb{R}^n \text{ and } t > 0\}.$$

Definition 1 Let G be a subset of $\mathcal{G}$; a function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be *lower G-semidifferentiable* at $x \in \mathbb{R}^n$ if there exists $\underline{D}_G f(x, \cdot) \in G$, called *lower G-semiderivative* of f at x, such that

$$\liminf_{v \to 0} \frac{f(x+v) - f(x) - \underline{D}_G f(x, v)}{\|v\|} = \ell_- \geq 0 \tag{1}$$

and for any other $g \in G$ satisfying (1) we have

$$\underline{D}_G f(x, v) \geq g(v), \qquad \forall v \in \mathbb{R}^n. \tag{2}$$

Any element $g \in G$ satisfying (1) is called *lower G-approximation* and ℓ_- is called the *order of lower G-approximation* of the function f at the point x.

The notion of *upper G-semidifferentiability* is defined quite similarly, replacing lim inf with lim sup and reversing the inequality direction in (1) and in (2). The function f is said to be *G-differentiable* if it is upper and lower G-semidifferentiable with respect to the same positively homogeneous function.

Example 1 Consider the set

$$G = \{g \in \mathcal{G} : g(v) \leq 0, \quad \text{for all } v \in \mathbb{R}\}$$

and the function $f : \mathbb{R} \to \mathbb{R}$ defined as $f(x) = |x| - x^2$. Since

$$\liminf_{v \to 0} \frac{|v| - v^2}{|v|} = 1$$

then $g(v) = 0$ is a lower G-approximation of f at $x = 0$. Moreover, since all the other elements of G are smaller than g then g is the lower G-semiderivative. The order of lower G-approximation is 1. Notice that f has not upper G-approximation since for each $g \in G$ we have

$$\limsup_{v \to 0} \frac{|v| - v^2 - g(v)}{|v|} = 1 - \min\{g(-1), g(1)\} \geq 1.$$

Example 2 Consider the set $G = \{g \in \mathcal{G} : g \text{ is convex }\}$ and the lipschitzian function $f : \mathbb{R} \to \mathbb{R}$ defined as

$$f(x) = \begin{cases} -x^2 \sin\left(\frac{1}{x}\right), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

First of all, since

$$\liminf_{v \to 0} \frac{-v^2 \sin\left(\frac{1}{v}\right)}{|v|} = 0$$

then $g = 0$ is a lower G-approximation. Moreover for any other positively homogeneous function g we have

$$\liminf_{v \to 0} \frac{-v^2 \sin\left(\frac{1}{v}\right) - g(v)}{|v|} = -\max\{g(-1), g(1)\};$$

then g is a lower G-approximation if and only if $\max\{g(-1), g(1)\} \leq 0$ and g is convex: this is satisfied by $g = 0$ only. Then f is lower G-semidifferentiable and the lower G-semiderivative is $\underline{D}_G f(0, v) = 0$. Notice that the classical Clarke directional derivative is $f^\circ(0, v) = |v|$.

2 A mean value result

The definition of semidifferentiability, recalled in the previous section, seems us very flexible thanks to the role of the set G. By fixing G, we can follow different aims. Nevertheless, in our opinion, there are some minimal properties that the elements of the set G must satisfy, otherwise a very poor analysis can be developed. In this spirit we fix the following two properties for the set G.
Property (B) Every $g \in G$ is bounded from below in a neighborhood of 0;
Property (N) For every $g \in G$ there exists a $k_0 > 0$ such that $g + k\|\cdot\| \in G$ for all $k \in [0, k_0)$.

Theorem 1 *Let $\varphi : \mathbb{R} \to \mathbb{R}$ be such that $\varphi(0) = \varphi(1)$ and suppose that $G \subseteq \mathcal{G}$ satisfies properties (B) and (N). If φ is lower G-semidifferentiable then there exists $c \in [0, 1)$ such that $\underline{D}_G\varphi(c, 1) \geq 0$.*

Proof Given $x \in \mathbb{R}^n$, since $\underline{D}_G\varphi(x, \cdot)$ satisfies property *(B)* we have

$$\liminf_{v\to 0} \underline{D}_G\varphi(x, v) = L > -\infty;$$

moreover the positive homogeneity implies that $L \geq 0$. Therefore, set

$$\varepsilon(x, v) = \varphi(x + v) - \varphi(x) - \underline{D}_G\varphi(x, v),$$

we have

$$\begin{aligned}
\liminf_{v\to 0} \varphi(x + v) &= \liminf_{v\to 0} \big(\varphi(x) + \underline{D}_G\varphi(x, v) + \varepsilon(x, v)\big) \\
&\geq \varphi(x) + \liminf_{v\to 0} \underline{D}_G\varphi(x, v) + \liminf_{v\to 0} \varepsilon(x, v) \\
&\geq \varphi(x)
\end{aligned}$$

which implies the lower semicontinuity of φ and hence the existence of a global minimum point $c \in [0, 1)$. Now we show that the order of lower G-approximation of φ is 0. By contradiction suppose that $\ell_- > 0$; therefore there exist $\alpha > 0$ and $r > 0$ such that, for each $x \in \mathbb{R}$,

$$\frac{\varphi(x + v) - \varphi(x) - \underline{D}_G\varphi(x, v)}{|v|} \geq \alpha, \qquad \forall v \in (-r, r) \setminus \{0\}$$

and then

$$\varphi(x + v) - \varphi(x) - \underline{D}_G\varphi(x, v) - \alpha|v| \geq 0, \qquad \forall v \in (-r, r) \setminus \{0\}.$$

Define $g(v) = \underline{D}_G\varphi(x, v) + \beta|v|$ where $2\beta = \min\{\alpha, k_0\}$. From the property *(N)* we have $g \in G$ and moreover g is a lower G-approximation bigger than $\underline{D}_G\varphi(x, \cdot)$ which contradicts the definition of lower G-semiderivative; therefore $\ell_- = 0$. For this reason we have

$$\liminf_{x\to c^+} \frac{\varphi(x) - \varphi(c)}{|x - c|} = \underline{D}_G\varphi(c, 1)$$

But c is a minimum point and this completes the proof. □

In order to deduce a mean value result, we introduce the following further directional property of the set G.

Property (D) for every $g \in G$ and for every $v \in \mathbb{R}^n$ there exists $k_0 > 0$ such that the positively homogeneous function

$$\widehat{g}(u) = \begin{cases} g(u) & \text{if } u \neq tv \\ g(u) + tk & \text{if } u = tv \end{cases}$$

belongs to G for every $k \in [0, k_0)$.

Theorem 2 *Suppose that $G \subseteq \mathcal{G}$ satisfies properties (B) and (D). Let $f : \mathbb{R}^n \to \mathbb{R}$ be lower G-semidifferentiable then, for all $x_1, x_2 \in \mathbb{R}^n$, there exists $x_0 \in [x_1, x_2)$ such that*

$$f(x_2) - f(x_1) \leq \underline{D}_G f(x_0, x_2 - x_1).$$

Proof Let $x_t = x_1 + t(x_2 - x_1)$ be any point on the segment $[x_1, x_2]$ and consider the function $\varphi : \mathbb{R} \to \mathbb{R}$ defined

$$\varphi(t) = f(x_t) + t(f(x_1) - f(x_2)).$$

Obviously $\varphi(0) = \varphi(1)$; moreover, choosing $g(s) = \underline{D}_G f(x_t, s(x_2 - x_1)) + s(f(x_1) - f(x_2))$, we have

$$\begin{aligned} &\liminf_{s \to 0} \frac{\varphi(t+s) - \varphi(t) - g(s)}{|s|} \\ &\quad = \liminf_{s \to 0} \frac{f(x_t + s(x_2 - x_1)) - f(x_t) - \underline{D}_G f(x_t, s(x_2 - x_1))}{|s|} \geq 0. \end{aligned}$$

Let

$$H = \{h : \mathbb{R} \to \mathbb{R} \ : \ \exists \alpha \in [0, 1) \text{ s.t. } h(s) = g(s) + \alpha|s|\}$$

be a subset of positively homogeneous functions satisfying properties *(B)* and *(N)*; moreover $g \in H$. Since all elements of H are bigger than g, then g is the lower H-semiderivative of φ if and only if $h(s) = g(s) + \alpha|s|$ is not a lower H-approximation for every $\alpha \in (0, 1)$ and this descends from Property *(D)*. In fact suppose that there exists $\alpha \in (0, 1)$ such that

$$\liminf_{s \to 0} \frac{\varphi(t+s) - \varphi(t) - g(s) - \alpha|s|}{|s|} \geq 0$$

then we deduce that

$$\liminf_{s \to 0} \frac{\varphi(t+s) - \varphi(t) - g(s)}{|s|} \geq \alpha$$

If we show that the right hand side of the last inequality is zero we conclude that g is the lower H-semiderivative. Suppose by contradiction that

$$\liminf_{s \to 0} \frac{\varphi(t+s) - \varphi(t) - g(s)}{|s|} = l > 0$$

then for every $\varepsilon \in (0, l)$ there exists $s_\varepsilon > 0$ such that

$$\varphi(t+s) - \varphi(t) - g(s) > (l - \varepsilon)|s| > 0, \qquad \forall s \in (-s_\varepsilon, s_\varepsilon) \setminus \{0\}.$$

Define the following positively homogeneous function

$$\widehat{g}(v) = \begin{cases} \underline{D}_G f(x_t, v) & \text{if } v \neq s(x_2 - x_1) \\ \underline{D}_G f(x_t, v) + (l - \varepsilon)|s| & \text{if } v = s(x_2 - x_1) \end{cases}$$

Since G satisfies property *(D)*, then $\widehat{g} \in G$, $\widehat{g} \geq \underline{D}_G f(x_t, \cdot)$ and

$$\liminf_{v \to 0} \frac{f(x_t + v) - f(x_t) - \widehat{g}(v)}{\|v\|} \geq 0$$

this contradicts the fact that $\underline{D}_G f(x_t, \cdot)$ is the lower G-semiderivative of f at x_t and therefore

$$\liminf_{s \to 0} \frac{\varphi(t+s) - \varphi(t) - g(s)}{|s|} = 0.$$

Therefore, from Theorem 1, we deduce that there exists $t_0 \in [0, 1)$ such that

$$0 \leq \underline{D}_H \varphi(t_0, 1) = \underline{D}_G f(x_{t_0}, x_2 - x_1) + f(x_1) - f(x_2)$$

and this completes the proof. □

3 Some applications

From the classical mean value theorem, many results descend. In particular it is possible to prove connections between derivatives and monotonicity and convexity of the function. Similar results can be obtained also in this general scheme. In fact, by means of the previous mean value result, we may deduce the following consequences.

Theorem 3 *Suppose that $G \subseteq \mathcal{G}$ satisfies properties (B) and (D). A lower G-semidifferentiable function f is lipschitzian with constant $L > 0$ if and only if for each $x \in \mathbb{R}^n$*

$$\underline{D}_G f(x, v) \leq L\|v\|, \qquad \forall v \in \mathbb{R}^n; \tag{3}$$

i.e. if and only if for each $x \in \mathbb{R}^n$ the lower G-semiderivative $\underline{D}_G f(x, \cdot)$ is bounded on the unit ball.

Proof Fixed $x_1, x_2 \in \mathbb{R}^n$, from (3) and Theorem 2 we deduce that there exists $x_0 \in [x_1, x_2)$ such that

$$f(x_2) - f(x_1) \leq \underline{D}_G f(x_0, x_2 - x_1) \leq L\|x_2 - x_1\|.$$

Changing the role of x_1 with x_2 we deduce that f is lipschitzian. For the converse, let $v \in \mathbb{R}^n$ be fixed with $\|v\| = 1$ and $\varepsilon > 0$. By the assumption of lower G-semidifferentiability, there exists a sequence $\{t_k\} \to 0^+$ such that

$$-\varepsilon \leq \frac{f(x + t_k v) - f(x) - \underline{D}_G f(x, t_k v)}{t_k} \leq L - \underline{D}_G f(x, v), \qquad \forall k \in \mathbb{N}$$

where the last inequality descends from the lipschitzianity of f. Since ε is arbitrary and $\underline{D}_G f(x, \cdot)$ is positively homogeneous, we deduce (3) which concludes the proof. □

The next result is related to the monotonicity of a lower G-semidifferentiable function. We recall that a convex and pointed cone C defines the partial ordering relation $\leq_C$ on $\mathbb{R}^n$ as

$$x_1 \leq_C x_2 \quad \Longleftrightarrow \quad x_2 - x_1 \in C.$$

A function f is called *C-decreasing* if $x_1 \leq_C x_2$ implies $f(x_1) \geq f(x_2)$.

Theorem 4 *Let C be a convex and pointed cone and suppose that $G \subseteq \mathcal{G}$ satisfies properties (B) and (D). Let $f : \mathbb{R}^n \to \mathbb{R}$ be lower G-semidifferentiable; if*

$$\underline{D}_G f(x, v) \leq 0, \qquad \forall v \in C$$

then f is C-decreasing.

Proof Suppose, by contradiction, that there exist $x_1, x_2 \in \mathbb{R}^n$ with $x_1 \leq_C x_2$ such that $f(x_1) < f(x_2)$. From Theorem 2 we deduce that there exists $x_0 \in [x_1, x_2)$ such that $\underline{D}_G f(x_0, x_2 - x_1) > 0$ that contradicts the assumption. □

The last result permits to characterize quasiconvex lower G-semidifferentiable functions by means of the quasimonotonicity of the lower G-semiderivative. We recall that

– a function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be *quasiconvex* if

$$f(tx_1 + (1-t)x_2) \leq \max\{f(x_1), f(x_2)\}$$

for any $x_1, x_2 \in \mathbb{R}^n$ and $t \in [0, 1]$;

– a bifunction $g : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is said to be *quasimonotone* if

$$\min\{g(x_1, x_2 - x_1), g(x_2, x_1 - x_2)\} \leq 0$$

for any $x_1, x_2 \in \mathbb{R}^n$.

Theorem 5 *Suppose that $G \subseteq \mathcal{G}$ satisfies the properties (B) and (D). A lower G-semidifferentiable function f is quasiconvex if and only if the bifunction $\underline{D}_G f$ is quasimonotone.*

Proof In [4] it was proved that the quasiconvexity of f implies the quasimonotonicity of the lower Dini directional derivative $D_- f$ defined as

$$D_- f(x, v) = \liminf_{t \to 0^+} \frac{f(x + tv) - f(x)}{t}.$$

Moreover we observe (see [1]) that the lower G-semidifferentiability of f implies that $\underline{D}_G f(x, v) \leq D_- f(x, v)$ and therefore the quasimonotonicity of $D_- f$ implies the quasimonotonicity of $\underline{D}_G f(x, v)$.

For the converse, suppose that f is not quasiconvex, that is there exist $a, b, c \in \mathbb{R}^n$ with $b \in (a, c)$ such that

$$f(b) > \max\{f(a), f(c)\}.$$

From Theorem 2 there exist $x_1 \in [a, b)$ and $x_2 \in [b, c)$ such that

$$\underline{D}_G f(x_1, b - a) > 0 \quad \text{and} \quad \underline{D}_G f(x_2, b - c) > 0.$$

By construction, there are $t_1, t_2 > 0$ such that $x_2 - x_1 = t_1(b - a)$ and $x_1 - x_2 = t_2(b - c)$ and therefore

$$\underline{D}_G f(x_1, x_2 - x_1) > 0 \quad \text{and} \quad \underline{D}_G f(x_2, x_1 - x_2) > 0$$

which show that $\underline{D}_G f$ is not quasimonotone. □

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